

Deriving the Logistic

$$\frac{dy}{dt} = ky \left(1 - \frac{y}{L}\right)$$

$$\frac{dy}{dt} = \frac{k}{L} y (L - y)$$

$$\int \frac{L}{y(L-y)} dy = k \int dt$$

Partial Fraction Decomposition:

$$\frac{L}{y(L-y)} = \frac{A}{y} + \frac{B}{L-y}$$

$$L = A(L-y) + B(y)$$

$$\text{set } y=0, AL=L, A=1$$

$$\text{set } y=L, BL=L, B=1$$

$$\int \frac{1}{y} + \frac{1}{L-y} dy = kt + C$$

$$\ln|y| - \ln|L-y| = kt + C$$

$$\ln\left(\frac{y}{L-y}\right) = kt + C$$

$$\frac{y}{L-y} = e^{kt+C} = \underbrace{(e^C)}_{\text{let } b = e^C} e^{kt}$$

$$\frac{L-y}{y} = b e^{-kt}$$

$$\frac{L}{y} - \frac{L}{L-y} = b e^{-kt}$$

$$\frac{L}{y} = 1 + b e^{-kt}$$

$$\frac{L}{y} = \frac{1}{1 + b e^{-kt}}$$

$$y = \frac{L}{1 + b e^{-kt}}$$